

Doubly elliptic strings on the (anti-)de Sitter manifold

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Abstract

We present a new class of elliptic-like strings on two-dimensional manifolds of constant curvature. Our solutions are related to a class of identities between Jacobi theta functions and to the geometry of the lightcone in one (spacelike) dimension more.

1 Introduction

The AdS/CFT correspondence has triggered a revival of interest in the classical Anti-de Sitter (AdS) string equations. Such equations have been for long known to be classically integrable on general grounds [1, 2, 3]. In the last decade however, several concrete explicit solutions have been constructed and mapped into properties of the dual conformal theory [4, 5] (for a review and a partial list of references see [3]). The various constructions often make use of ingenious procedures, as for instance the one introduced by Pohlmeyer [1], to transform the string equations and the constraints into other kind of equations for which integration methods already exist, such as the sine-Gordon equation, etc.

In this paper we present a direct construction of certain doubly elliptic solutions (i.e. solution which are elliptic functions of both the worldsheet coordinates t and s) whose intriguing properties seem to have stayed uncovered to date. These solutions can be obtained in many different ways. Here we present a construction based on the embedding of the AdS space into the projective cone in one dimension more. We exhibit explicit parameterizations for the two-dimensional anti de Sitter, de Sitter and Lobachevskij strings, while we leave the higher dimensional hyperelliptic cases for future investigation [6].

We begin by reformulating the GKP solution [4] on the projective cone in one spacelike dimension more. This mapping put into evidence the deep relation that exists between AdS strings (and more generally strings on complex spheres) and theta functions (and more generally hyperelliptic functions). A particular role is played by the Virasoro constraints that are spelled by a class of (perhaps unknown) beautifully simple identities between theta functions and their derivatives. Based on these identities we present several interesting new string solutions. In a related paper we will discuss the general issue of separating the of variables in the AdS classical strings equations with application to nontrivial examples in higher dimension [6].

2 Summary of the GKP's rotating string.

Even though very well known, to motivate what follows we start by briefly reviewing the GKP rigidly rotating folded string. One considers here a two-dimensional parameterized surface embedded in AdS_3 as follows:

$$\begin{cases} Y_0 = \cosh \rho(s) \cos(\omega_1 t), & Y_1 = \cosh \rho(s) \sin(\omega_1 t), \\ Y_2 = \sinh \rho(s) \cos(\omega_2 t), & Y_3 = \sinh \rho(s) \sin(\omega_2 t), \end{cases} \quad (1)$$

where

$$AdS_d = \{Y \in \mathbb{R}^{d+1} : Y^2 = Y \cdot Y = Y_0^2 + Y_1^2 - Y_2^2 - Y_3^2 - \dots - Y_d^2 = 1\}. \quad (2)$$

The function ρ is required to be periodic: $\rho(s) = \rho(s + 2L)$ where L is an adjustable parameter.

The second step is to derive a differential equation for $\rho(s)$ by imposing the conformal gauge constraints:

$$(\partial_t Y \pm \partial_s Y)^2 = \omega_1^2 \cosh^2 \rho - \omega_2^2 \sinh^2 \rho - (\rho')^2 = 0. \quad (3)$$

Eq. (3) immediately establishes a relation between the angular velocities ω_1 and ω_2 and the parameter L as follows:

$$2L = \int_0^{2L} ds = 4 \int_0^{\rho_0} \frac{d\rho}{\sqrt{\omega_1^2 \cosh^2 \rho - \omega_2^2 \sinh^2 \rho}} = 4 \int_0^k \frac{dx}{\omega_2 \sqrt{1-x^2} \sqrt{k^2-x^2}} = \frac{4K(k)}{\omega_2} \quad (4)$$

where $K(k)$ is the complete elliptic integral of the first kind and

$$\tanh \rho_0 = \pm \frac{\omega_1}{\omega_2} = \pm k$$

gives the localization of the two extremal points of the (closed) folded string.

A straightforward computation shows that any function $\rho(\sigma)$ solving the constraint (3) automatically provides a solution of the string equations

$$\partial_t^2 Y_i - \partial_s^2 Y_i + [(\partial_t Y)^2 - (\partial_s Y)^2] Y_i = 0 \quad (5)$$

through the embedding (1). Two important conserved quantities are easily computed in terms of the complete elliptic integrals:

$$\mathcal{E} = \int_0^{2L} (\dot{Y}_0 Y_1 - Y_0 \dot{Y}_1) d\sigma = \frac{4kE(k)}{1-k^2} \quad (6)$$

$$S = \int_0^{2L} (\dot{Y}_2 Y_3 - Y_2 \dot{Y}_3) d\sigma = \frac{4E(k)}{1-k^2} - 4K(k). \quad (7)$$

where $E(k)$ is the complete elliptic integral of the second kind. The above equations provide a relation $\mathcal{E} = \mathcal{E}(S)$ between $\mathcal{E}(k)$ and $S(k)$ (in parametric form) which is relevant to compute the anomalous dimension of twist operators in the dual conformal field theory [4, 7].

To integrate Eq. (3) we pose $y = \cosh \rho$. Eq. (3) is transformed into the nonlinear differential equation satisfied by a certain Jacobian elliptic function:

$$(y')^2 = \omega_2^2 [-1 + (2-k^2)y^2 - (1-k^2)y^4]. \quad (8)$$

The natural initial condition $\rho(0) = 0$ gives rise to the solution

$$\cosh \rho = \text{nd}(\omega s; k), \quad \sinh \rho = k \text{sd}(\omega s; k), \quad (9)$$

where we set $\omega_2 = \omega$; the string worldsheet is finally parameterized as follows:

$$\begin{cases} Y_0 = \text{nd}(\omega s; k) \cos(k\omega t), & Y_1 = \text{nd}(\omega s; k) \sin(k\omega t), \\ Y_2 = k \text{sd}(\omega s; k) \cos(\omega t), & Y_3 = k \text{sd}(\omega s; k) \sin(\omega t). \end{cases} \quad (10)$$

A second simple solution is obtained from the latter by the quarter period shift $\omega s \rightarrow \omega s + K$:

$$\cosh \rho = \frac{1}{k'} \text{dn}(\omega s; k), \quad \sinh \rho = \frac{k}{k'} \text{cn}(\omega s; k). \quad (11)$$

where $k' = \sqrt{1-k^2}$ is the complementary modulus. This completes our review of the GKP solution.

2.1 Rotating strings on AdS_5

An immediate generalization of the original GKP rigidly rotating string may be obtained by the following ansatz:

$$\begin{cases} Y_0 = b \operatorname{dn}(\kappa s, k) \cos(\omega t), & Y_1 = b \operatorname{dn}(\kappa s, k) \sin(\omega t), \\ Y_2 = b_1 \operatorname{sn}(\kappa s, k) \cos(\omega_1 t), & Y_3 = b_1 \operatorname{sn}(\kappa s, k) \sin(\omega_1 t), \\ Y_4 = b_2 \operatorname{cn}(\kappa s, k) \cos(\omega_2 t), & Y_5 = b_2 \operatorname{cn}(\kappa s, k) \sin(\omega_2 t). \end{cases} \quad (12)$$

Two conditions on the three coefficients b follow by imposing $Y \cdot Y = 1$ (i.e. $Y \in AdS_5$):

$$b_1^2 = k'^2 b^2 - 1, \quad b_2^2 = b^2 - 1. \quad (13)$$

Next, we impose the Virasoro constraints $(\partial_t Y \pm \partial_s Y)^2 = 0$ and get two more conditions on the angular velocities:

$$\omega_1^2 = \frac{k'^2 ((b^2 - 1) \kappa^2 - b^2 \omega^2)}{1 - k'^2 b^2}, \quad \omega_2^2 = \frac{(1 - k'^2 b^2) \kappa^2}{b^2 - 1} + \frac{b^2 \omega^2}{b^2 - 1}. \quad (14)$$

Finally, the string equations (5) give one more condition (at variance with the original GKP model where the constraints are enough for the string equation to hold, because of its lower dimensionality):

$$b^2 = \frac{1}{2k'^2} \left(\frac{\omega^2}{\kappa^2} + k'^2 + 1 \right) \quad (15)$$

which together with Eqs. (13) and (14) implies that

$$\omega_1^2 = \omega^2 + \kappa^2, \quad \omega_2^2 = \omega^2 + k'^2 \kappa^2, \quad (16)$$

$$b_1^2 = \frac{1}{2} \left(\frac{\omega^2}{\kappa^2} - k^2 \right), \quad b_2^2 = \frac{1}{2k'^2} \left(\frac{\omega^2}{\kappa^2} + k^2 \right). \quad (17)$$

It is immediately seen that the choice $\omega^2 = k^2 \kappa^2$ reproduces the standard GKP solution as given in Eq. (11). On the other hand, for any value of the elliptic modulus k there is a one parameter family of rotating GKP-like strings parameterized by an angular velocity $\omega \geq k\kappa$. Now we have three non-zero Cartan conserved quantities, the energy and two spins (we set $\kappa = 1$):

$$\mathcal{E}_2 = \frac{2\omega}{k'^2} (\omega^2 + k'^2 + 1) E(k^2), \quad (18)$$

$$S_1 = \frac{2}{k^2} (\omega^2 - k^2) \sqrt{(\omega^2 + 1)} (K(k^2) - E(k^2)), \quad (19)$$

$$S_2 = \frac{2}{k^2 k'^2} (\omega^2 + k^2) \sqrt{\omega^2 + k'^2} (E(k^2) - k'^2 K(k^2)), \quad (20)$$

which are related by a law $\mathcal{E}_2 = \mathcal{E}_2(S_1, S_2)$ that in principle can be obtained by eliminating ω and k ; for $S_1 = 0$ the function $\mathcal{E}_2(S_1, S_2)$ reproduces the energy $\mathcal{E}(S)$ of the standard GKP string:

$$\mathcal{E} = \mathcal{E}(S) = \mathcal{E}_2(0, S) \quad (21)$$

3 The GKP string in homogeneous coordinates

There is a nice geometrical reinterpretation of the above construction that can be uncovered by recasting the parametrization (10) of the GKP string in terms of theta functions (we adopt the notations of Whittaker and Watson's classical book [8]). For the reader's convenience we recall the basic formulae expressing the Jacobi elliptic functions as ratios of theta functions and theta constants:

$$\operatorname{sn}(z, k) = \frac{\vartheta_3 \vartheta_1(z/\vartheta_3^2)}{\vartheta_2 \vartheta_4(z/\vartheta_3^2)}, \quad \operatorname{cn}(z, k) = \frac{\vartheta_4 \vartheta_2(z/\vartheta_3^2)}{\vartheta_2 \vartheta_4(z/\vartheta_3^2)}, \quad \operatorname{dn}(z, k) = \frac{\vartheta_4 \vartheta_3(z/\vartheta_3^2)}{\vartheta_3 \vartheta_4(z/\vartheta_3^2)}. \quad (22)$$

The elliptic moduli k and k' are related to the theta constants as follows:

$$k = \frac{\vartheta_2^2}{\vartheta_3^2} = \frac{\vartheta_2^2(0|\tau)}{\vartheta_3^2(0|\tau)}, \quad k' = \frac{\vartheta_4^2}{\vartheta_3^2} = \frac{\vartheta_4^2(0|\tau)}{\vartheta_3^2(0|\tau)}. \quad (23)$$

The parametrization (10) of the worldsheet of the GKP string can then be rewritten as follows:

$$\begin{cases} Y_0 = \frac{\vartheta_3 \vartheta_4(\hat{s})}{\vartheta_4 \vartheta_3(\hat{s})} \cos(kt), & Y_1 = \frac{\vartheta_3 \vartheta_4(\hat{s})}{\vartheta_4 \vartheta_3(\hat{s})} \sin(kt), \\ Y_2 = \frac{\vartheta_2 \vartheta_1(\hat{s})}{\vartheta_4 \vartheta_3(\hat{s})} \cos(t), & Y_3 = \frac{\vartheta_2 \vartheta_1(\hat{s})}{\vartheta_4 \vartheta_3(\hat{s})} \sin(t), \end{cases} \quad (24)$$

where we set $\omega = 1$ and $\hat{s} = s/\vartheta_3^2$. The above parametrization naturally suggests the introduction of homogeneous coordinates on the cone

$$C_{2,d} = \{\xi \in \mathbb{R}^{d+2} : \xi^2 = \xi \cdot \xi = \xi_0^2 + \xi_1^2 - \xi_2^2 - \xi_3^2 - \dots - \xi_{d+1}^2 = 0\} \quad (25)$$

(here $d = 3$); there we consider the parameterized two-surface

$$\begin{cases} \xi_0 = \vartheta_3 \vartheta_4(\hat{s}) \cos(kt), & \xi_1 = \vartheta_3 \vartheta_4(\hat{s}) \sin(kt), \\ \xi_2 = \vartheta_2 \vartheta_1(\hat{s}) \cos(t), & \xi_3 = \vartheta_2 \vartheta_1(\hat{s}) \sin(t), \\ \xi_4 = \vartheta_4 \vartheta_3(\hat{s}); \end{cases} \quad (26)$$

the string (10) on the anti de Sitter GKP is reobtained by taking the ratios $Y_i = \xi_i/\xi_{d+1}$. Note that the condition $\xi \in C_{2,d}$ just expresses a well-known quadratic identity between theta functions ([8], p. 466):

$$\xi^2 = \vartheta_3^2 \vartheta_4(\hat{s})^2 - \vartheta_2^2 \vartheta_1(\hat{s})^2 - \vartheta_4^2 \vartheta_3(\hat{s})^2 = 0. \quad (27)$$

The second observation is that the parameterized surface (26) obeys Virasoro-type constraints on the cone. The validity of the constraint $\partial_t \xi \cdot \partial_s \xi = 0$ is immediate. The nontrivial constraint can be shown by using the heat equation which is satisfied by the theta functions (see e.g. [8]). Indeed one has that

$$\begin{aligned} \partial_s \xi \cdot \partial_s \xi &= \frac{\omega^2}{\vartheta_3^4} (\vartheta_3^2 \vartheta_4'(\hat{s})^2 - \vartheta_2^2 \vartheta_1'(\hat{s})^2 - \vartheta_4^2 \vartheta_3'(\hat{s})^2) = \\ &= \frac{4\omega^2}{\pi i \vartheta_3^4} \left(\frac{\partial \vartheta_3(0|\tau)}{\partial \tau} \vartheta_3 \vartheta_4(\hat{s})^2 - \frac{\partial \vartheta_2(0|\tau)}{\partial \tau} \vartheta_2 \vartheta_1(\hat{s})^2 - \frac{\partial \vartheta_4(0|\tau)}{\partial \tau} \vartheta_4 \vartheta_3(\hat{s})^2 \right). \end{aligned} \quad (28)$$

This equality implies that the function $\partial_s \xi(\cdot) \cdot \partial_s \xi(\cdot)$ is quasi doubly periodic function. By further observing that $\partial_s \xi(\frac{\pi}{2}) \cdot \partial_s \xi(\frac{\pi}{2}) = 0$ one infers that

$$\partial_s \xi \cdot \partial_s \xi = \frac{\omega^2}{\vartheta_3^4} (\vartheta_3^2 \vartheta_4'(\hat{s})^2 - \vartheta_2^2 \vartheta_1'(\hat{s})^2 - \vartheta_4^2 \vartheta_3'(\hat{s})^2) = -\omega^2 \frac{\vartheta_2^2 \vartheta_4^2}{\vartheta_3^2} \vartheta_2(\hat{s})^2. \quad (29)$$

On the other hand

$$\partial_t \xi \cdot \partial_t \xi = \omega^2 \frac{\vartheta_2^2}{\vartheta_3^2} (\vartheta_2^2 \vartheta_4(\hat{s})^2 - \vartheta_3^2 \vartheta_1(\hat{s})^2) = \omega^2 \frac{\vartheta_2^2 \vartheta_4^2}{\vartheta_3^2} \vartheta_2(\hat{s})^2 \quad (30)$$

and therefore the constraint $\partial_t \xi \cdot \partial_t \xi + \partial_s \xi \cdot \partial_s \xi = 0$ is satisfied.

The above result is indeed generally true in the following sense: suppose that an AdS string is parameterized in terms of inhomogeneous coordinates as follows

$$t, s \rightarrow Y_i(t, s) = \frac{\xi_i(t, s)}{\xi_{d+1}(t, s)}, \quad i = 0, 1, \dots, d; \quad (31)$$

the map $t, s \rightarrow \xi_\mu(t, s)$, $\mu = 0, 1, \dots, d+1$, describes a two-surface in $C_{2,d}$. The condition $\xi^2 = 0$ immediately implies that

$$\sum_{i=0}^{d-1} \partial_z Y_i \partial_w Y_i = \frac{1}{\xi_{d+1}^2} \sum_{\mu=0}^d \partial_z \xi_\mu \partial_w \xi_\mu, \quad (32)$$

where z, w can be either t or s . Therefore, if the collection of functions Y_i satisfies the constraints in AdS_d , the functions ξ_μ also do in $C(2, d)$ and viceversa. This also means that the nontrivial identity (29) among theta functions and their derivatives is indeed *proven* by exhibiting the map (26), once the validity of the Virasoro constraints for the corresponding AdS string is known.

4 Doubly elliptic solutions - Preliminaries.

We are now ready to construct new examples of AdS classical strings. Here the idea of rotating strings is abandoned in favor of solutions which are elliptic functions of both the worldsheet coordinates t and s . These strings have a completely different shape w.r.t. the rotating examples and may have interesting features also in view of generalizations to higher dimensions.

Let us describe the typical construction in some detail and consider the following symmetric two-surface embedded in the cone $C_{2,2}$:

$$(t, s) \rightarrow \begin{cases} \xi_0 = \vartheta_1(\hat{t}) \vartheta_1(\hat{s}), & \xi_1 = \vartheta_3(\hat{t}) \vartheta_3(\hat{s}), \\ \xi_2 = \vartheta_2(\hat{t}) \vartheta_2(\hat{s}), & \xi_3 = \vartheta_4(\hat{t}) \vartheta_4(\hat{s}). \end{cases} \quad (33)$$

The vector ξ satisfies the condition $\xi^2 = 0$ (i.e. $\xi \in C_{2,2}$) because of a well-known quadratic identity between theta functions of different arguments ([8], p 487):

$$\xi^2 = \vartheta_1(\hat{t})^2 \vartheta_1(\hat{s})^2 - \vartheta_2(\hat{t})^2 \vartheta_2(\hat{s})^2 + \vartheta_3(\hat{t})^2 \vartheta_3(\hat{s})^2 - \vartheta_4(\hat{t})^2 \vartheta_4(\hat{s})^2 = 0. \quad (34)$$

The symmetry $\xi_\mu(t, s) = \xi_\mu(s, t)$ will reverberate in a certain self-duality of the corresponding strings (we will come back to this point below). Let us examine the status of the Virasoro constraints:

$$\partial_t \xi \cdot \partial_t \xi + \partial_s \xi \cdot \partial_s \xi = 0, \quad \partial_t \xi \cdot \partial_s \xi = 0. \quad (35)$$

The validity of the first constraint amounts to establishing another (possibly unknown) identity among theta functions and their derivatives:

$$\partial_t \xi \cdot \partial_t \xi + \partial_s \xi \cdot \partial_s \xi = -\theta_3^{-2} \sum_{i=1}^4 (-1)^\alpha (\vartheta'_\alpha(\hat{t})^2 \vartheta_\alpha(\hat{s})^2 + \vartheta_\alpha(\hat{t})^2 \vartheta'_\alpha(\hat{s})^2) = 0. \quad (36)$$

This identity may be proven by applying the Laplace operator to (34):

$$\begin{aligned} 0 &= \frac{1}{2} (\partial_x^2 + \partial_y^2) \sum_{\alpha=1}^4 (-1)^\alpha (\vartheta_\alpha(x)^2 \vartheta_\alpha(y)^2) = \\ &= \sum_{\alpha=1}^4 (-1)^\alpha (\vartheta'_\alpha(x)^2 \vartheta_\alpha(y)^2 + \vartheta_\alpha(x)^2 \vartheta'_\alpha(y)^2 + \vartheta_\alpha(x) \vartheta''_\alpha(x) \vartheta_\alpha(y)^2 + \vartheta_\alpha(x)^2 \vartheta_\alpha(y) \vartheta''_\alpha(y)) = \\ &= \sum_{\alpha=1}^4 (-1)^\alpha (\vartheta'_\alpha(x)^2 \vartheta_\alpha(y)^2 + \vartheta_\alpha(x)^2 \vartheta'_\alpha(y)^2) + \frac{2i}{\pi} \frac{\partial}{\partial \tau} \sum_{\alpha=1}^4 (-1)^\alpha (\vartheta_\alpha(x)^2 \vartheta_\alpha(y)^2) = \\ &= \sum_{\alpha=1}^4 (-1)^\alpha (\vartheta'_\alpha(x)^2 \vartheta_\alpha(y)^2 + \vartheta_\alpha(x)^2 \vartheta'_\alpha(y)^2). \end{aligned} \quad (37)$$

In the second step we used once more the heat equation. The other constraint $\partial_t \xi \cdot \partial_s \xi = 0$ is an immediate consequence of the separation of variables in (33).

We can take one step further in understanding Eq. (36) by explicitly computing the quantity

$$\sum_{\alpha=1}^4 (-1)^\alpha \vartheta_\alpha(x)^2 \vartheta'_\alpha(y)^2. \quad (38)$$

By Taylor expanding the Jacobi addition formulae ([8], p 487)

$$\vartheta_4^2 \vartheta_\alpha(y+z) \vartheta_\alpha(y-z) = \vartheta_\alpha(y)^2 \vartheta_4(z)^2 - \vartheta_{5-\alpha}(y)^2 \vartheta_1(z)^2, \quad \alpha = 1, 2, 3, 4. \quad (39)$$

we rapidly deduce the following differential identities:

$$\vartheta'_\alpha(y)^2 = \frac{2i}{\pi} \frac{\partial}{\partial \tau} \vartheta_\alpha(y)^2 - \vartheta_\alpha(y)^2 \frac{\vartheta''_4}{\vartheta_4} + \vartheta_{5-\alpha}(y)^2 \left(\frac{\vartheta'_1}{\vartheta_4} \right)^2. \quad (40)$$

By multiplying Eq. (40) by $(-1)^\alpha \vartheta_\alpha(x)^2$ (where x is an independent variable) and summing over the α 's, it follows that

$$\sum_{\alpha=1}^4 (-1)^\alpha \vartheta_\alpha(x)^2 \vartheta'_\alpha(y)^2 = \frac{2i}{\pi} \sum_{\alpha=1}^4 (-1)^\alpha \vartheta_\alpha(x)^2 \frac{\partial}{\partial \tau} \vartheta_\alpha(y)^2 + \sum_{\alpha=1}^4 (-1)^\alpha \vartheta_\alpha(x)^2 \vartheta_{5-\alpha}(y)^2 \left(\frac{\vartheta'_1}{\vartheta_4} \right)^2. \quad (41)$$

On the other hand

$$\frac{1}{2} \partial_y^2 \sum_{\alpha=1}^4 (-1)^\alpha (\vartheta_\alpha(x)^2 \vartheta_\alpha(y)^2) = 0 = \sum_{\alpha=1}^4 (-1)^\alpha \vartheta_\alpha(x)^2 \vartheta'_\alpha(y)^2 + \frac{2i}{\pi} \sum_{\alpha=1}^4 (-1)^\alpha \vartheta_\alpha(x)^2 \frac{\partial}{\partial \tau} (\vartheta_\alpha(y)^2) \quad (42)$$

Putting everything together we finally get the formula

$$\sum_{\alpha=1}^4 (-1)^\alpha \vartheta_\alpha(x)^2 \vartheta'_\alpha(y)^2 = (\vartheta_4(x)^2 \vartheta_1(y)^2 - \vartheta_1(x)^2 \vartheta_4(y)^2) (\vartheta_2 \vartheta_3)^2 \quad (43)$$

where the antisymmetry of the expression (38) is manifest and the validity of the constraint equation (36) is confirmed.

5 Finite open strings.

Let us now project back the surface (33) on the anti de Sitter manifold by singling out at first the fourth coordinate $\xi_3(\hat{t}, \hat{s})$. The map

$$(t, s) \rightarrow Y^{(1)} = \begin{cases} Y_0(t, s) &= \frac{\xi_0}{\xi_3} = \frac{\vartheta_1(\hat{t})\vartheta_1(\hat{s})}{\vartheta_4(\hat{t})\vartheta_4(\hat{s})} = k \operatorname{sn}(t, k) \operatorname{sn}(s, k), \\ Y_1(t, s) &= \frac{\xi_1}{\xi_3} = \frac{\vartheta_3(\hat{t})\vartheta_3(\hat{s})}{\vartheta_4(\hat{t})\vartheta_4(\hat{s})} = \frac{1}{k'} \operatorname{dn}(t, k) \operatorname{dn}(s, k), \\ Y_2(t, s) &= \frac{\xi_2}{\xi_3} = \frac{\vartheta_2(\hat{t})\vartheta_2(\hat{s})}{\vartheta_4(\hat{t})\vartheta_4(\hat{s})} = \frac{k}{k'} \operatorname{cn}(t, k) \operatorname{cn}(s, k), \end{cases} \quad (44)$$

is an embedding of a two-surface in the two-dimensional anti-de Sitter manifold AdS_2 (see Fig. 1). Eq. (44) represents equally well a world-sheet on the two-dimensional de Sitter universe dS_2 , the two manifolds coinciding in dimension two. In both cases the Virasoro constraints are satisfied by Eqs. (32) and (36). As for the GKP string, the validity of the constraints is enough for the string equations to hold (this fact is of course not true in general dimension). We can elaborate on this as follows: a concise way of writing the second derivatives of the Y_i 's is

$$\partial_z^2 Y_i = (2k^2 \operatorname{sn}(z, k)^2 - \kappa_i^2) Y_i \quad (45)$$

where z denotes either t or s and $\kappa_0^2 = 1 + k^2$, $\kappa_1^2 = 1$, $\kappa_2^2 = k^2$. From Eqs. (32) and (43) we get that

$$(\partial_t Y)^2 = \frac{(\vartheta_1(\hat{s})^2 \vartheta_4(\hat{t})^2 - \vartheta_4(\hat{s})^2 \vartheta_1(\hat{t})^2) (\vartheta_2/\vartheta_3)^2}{\vartheta_4^2(\hat{t}) \vartheta_4^2(\hat{s})} = k^2 \left(\operatorname{sn}(s, k)^2 - \operatorname{sn}(t, k)^2 \right) = -(\partial_s Y)^2 \quad (46)$$

so that

$$\partial_z^2 Y_i + (\partial_z Y)^2 Y_i = \left(k^2 \operatorname{sn}(s, k)^2 + k^2 \operatorname{sn}(t, k)^2 - \kappa_i^2 \right) Y_i. \quad (47)$$

The symmetry of the rhs in the exchange of t and s finally shows that the string equations (5) are satisfied. The worldsheet of the string is a compact surface symmetric by reflection w.r.t. the two arcs of geodesics joining its four vertices. The worldsheet is bordered by the four light rays joining the same vertices (see Figures 1 and 2). A perhaps surprising feature of the above symmetric solution is that the family of curves $s \rightarrow Y(t, s)$ (labeled by fixed values of t) and the family of curves $t \rightarrow Y(t, s)$ (labeled by fixed values of s) coincide. Any such curve is a closed (i.e periodic) curve made by four arcs, two spacelike and two timelike; the four distinguished points where the tangent to the curve becomes

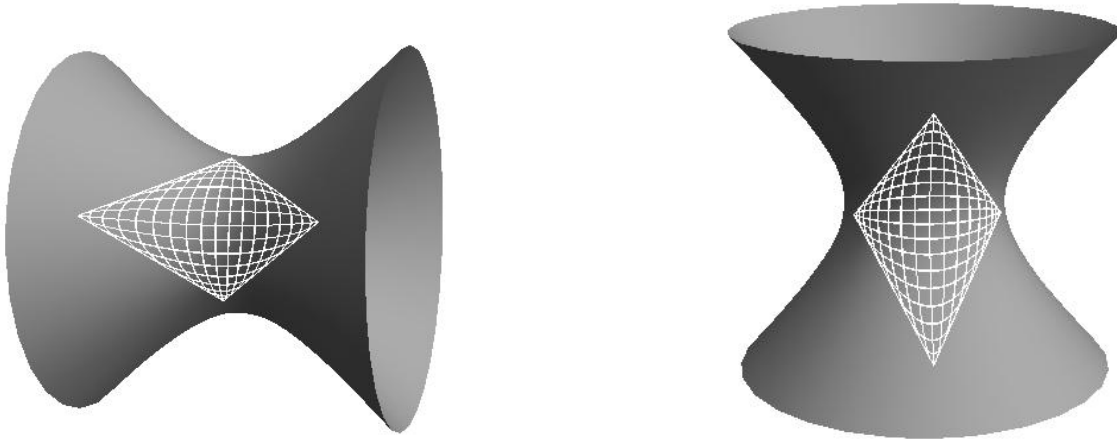


Figure 1: A classical string on the anti de Sitter (resp. de Sitter) manifold, being created at event, reaching its maximal extension, recontracting and being annihilated at a final event.

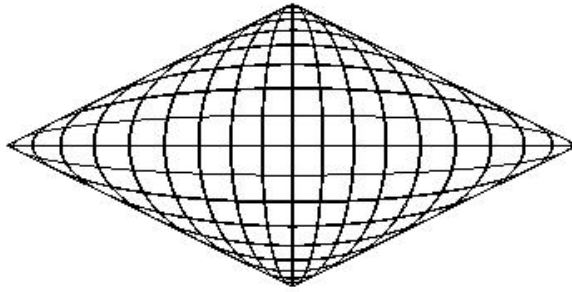


Figure 2: The compact world-surface of an anti de Sitter classical string (44) bordered by four light rays. The diagonals are geodesics.

lightlike move at the speed of light and may be interpreted as the string endpoints (see Figure 2). By Eq. (46) those endpoints are identified by the conditions

$$t = s, \quad 2K(k) - s, \quad 2K(k) + s, \quad 4K(k) - s, \quad (48)$$

where the elliptic integral $K(k)$ is the quarter period. A possible physical interpretation of this solution is that of an open string created at a certain event of the AdS spacetime and expanding up to a maximal length before starting to recontract. The configuration of maximal length (the diagonal) is the only one that coincides with a spacelike geodesic. At this moment the endpoints suddenly invert their motion and the string starts recontracting to be finally annihilated at an event which is the mirror image of the event where the string was created w.r.t the plane containing the diagonal spacelike geodesic (and the geometrical center of the AdS manifold). The midpoint of the string is at rest (i.e moves along a timelike geodesic) and we may use it to measure the proper lifetime of the string

$$t(k) = 2 \arcsin k = \arccos(1 - 2k^2) \quad (49)$$

The maximal length reached by the string is given by

$$l(k) = \operatorname{arccosh} \left(\frac{1 + k^2}{1 - k^2} \right). \quad (50)$$

The action – the area of the worldsheet – has an intrinsic geometric meaning:

$$\begin{aligned}\mathcal{A} &= \int \sqrt{h} dt ds = k^2 \int \sqrt{\left(\text{sn}(t, k)^2 - \text{sn}(s, k)^2\right)^2} dt ds = \\ &= 8k^2 \int_0^{2K} t \text{sn}(t, k)^2 dt - 8(K(k) - E(k))K(k) = 8(K(k) - E(k))K(k).\end{aligned}\quad (51)$$

A second possible interpretation of this solution is that of a folded closed string. Suppose indeed that there are two sheets; each time that the string hits the lightlike boundary passes to the other sheet and it closes back after a full period. The other sheet may thought as copies of the same AdS universe. In this case when passing to the second sheet the curve become timelike and then again spacelike and once more timelike before closing. Or either, it also is conceivable that the second sheet is a de Sitter one. In this case the four arcs are all spacelike and there are four points where the tangent to curve become lightlike. In both cases the following (folded) surface may be ascribed to the string:

$$\mathcal{A}' = k^2 \int \left(\text{sn}(t, k)^2 - \text{sn}(s, k)^2\right) dt \wedge ds = 16k^2 \int_0^K \text{sn}(t, k)^2 dt \int_0^K ds = 16(K(k) - E(k))K(k) \quad (52)$$

and obviously $\mathcal{A}' = 2\mathcal{A}$.

When

$$k = k_n = \sin(\pi/n), \quad n = 2, 3, \dots \quad (53)$$

the above process can be repeated to produce an oscillating string going from zero to its maximal extension and back forever on the anti de Sitter periodic manifold. If we drop the above quantization condition we may always consider a string oscillating forever but we have to move on the covering of the AdS manifold.

The right way is however to continue the solution (44) with a second one obtained by the substitution $k \rightarrow k'$. The so-obtained string is a time periodic solution on the true AdS manifold (as opposed to its covering - see Figure 3). The total action corresponding to this string is now given by

$$\mathcal{A} = \int \sqrt{h} dt ds = 8[K(k)^2 + K(k')^2 - E(k)K(k) - E(k')K(k')]. \quad (54)$$

6 Semi-infinite strings

Let us now single out the third coordinate $\xi_2(\hat{t}, \hat{s})$; we get another parameterized two-surface on AdS_2 (or dS_2), always satisfying the Virasoro constraints:

$$(t, s) \rightarrow Y^{(2)}(t, s; k) = \begin{cases} Y_0(t, s) &= \frac{\xi_0}{\xi_2} = \frac{\vartheta_1(\hat{t})\vartheta_1(\hat{s})}{\vartheta_2(\hat{t})\vartheta_2(\hat{s})} = k' \text{sc}(t, k) \text{sc}(s, k), \\ Y_1(t, s) &= \frac{\xi_1}{\xi_2} = \frac{\vartheta_3(\hat{t})\vartheta_3(\hat{s})}{\vartheta_2(\hat{t})\vartheta_2(\hat{s})} = \frac{1}{k} \text{dc}(t, k) \text{dc}(s, k), \\ Y_2(t, s) &= \frac{\xi_3}{\xi_2} = \frac{\vartheta_4(\hat{t})\vartheta_4(\hat{s})}{\vartheta_2(\hat{t})\vartheta_2(\hat{s})} = \frac{k'}{k} \text{nc}(t, k) \text{nc}(s, k); \end{cases} \quad (55)$$

The second derivatives of the Y_i 's have again a simple structure:

$$\partial_z^2 Y_i = (2 \text{dc}(z, k)^2 - \kappa_i^2) Y_i \quad (56)$$

where z denotes either s or t and $\kappa_0^2 = k^2$, $\kappa_1^2 = 1 + k^2$, $\kappa_2^2 = 1$. From Eqs. (32) and (43)

$$(\partial_t Y)^2 = -(\partial_s Y)^2 = \text{dc}(s, k)^2 - \text{dc}(t, k)^2. \quad (57)$$

and therefore

$$\partial_z^2 Y_i + (\partial_z Y)^2 Y_i = \left(\text{dc}(s, k)^2 + \text{dc}(t, k)^2 - \kappa_i^2\right) Y_i. \quad (58)$$

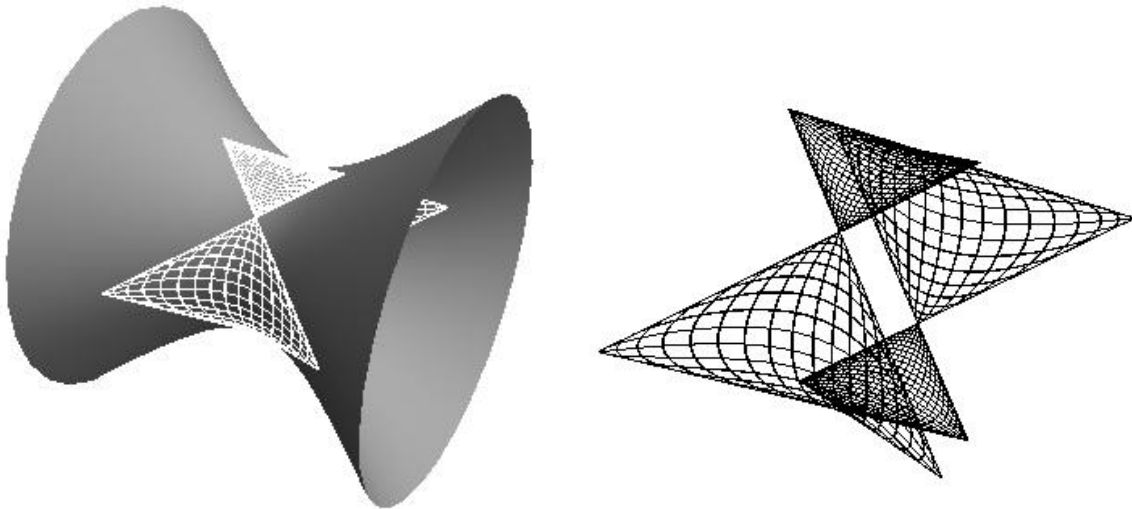


Figure 3: An oscillating periodic AdS solution. every time the string reaches a conical point the elliptic modulus jump to the complementary one. The boundary of the crown-shaped string worldsheet is made by eight lightlike vectors belonging to AdS bulk.

Once again the symmetry of the rhs in the exchange of t and s shows that that (55) solves the string equations (5) on the two-dimensional anti-de Sitter spacetime, or well on the two-dimensional de Sitter spacetime. The family of curves $s \rightarrow Y(t, s)$ (labeled by fixed values of t) and the family of curves $t \rightarrow Y(t, s)$ (labeled by fixed values of s) coincide. Any such curve is an infinite curve made by three arcs: two spacelike and one timelike in the AdS case; two timelike and one spacelike in the dS case; the two distinguished points where the tangent to the curve becomes lightlike move at the speed of light and may be interpreted as the string endpoints.

A possible physical interpretation of the string is as follows: in the anti de Sitter case we may think of a semi-infinite open string. One endpoint is at spacelike infinity, the other moves at the speed of light towards an event A. When the string configuration coincides with a spacelike geodesics, the second endpoint suddenly inverts the direction of its speed and the string goes away to infinity (see Figure 4). The de Sitter interpretation of this solution is that of a finite open string being created at an event A and expanding forever to timelike infinity (or well a string coming in from minus infinity to be annihilated at the event A).

The strings (44) and (55) can be merged into a single string made by three disconnected parts that touch at the precise moment where the endpoints invert their speed. As before the natural choice is to glue together the solution (44) at the elliptic modulus k with the solution (55) at k' . Indeed such strings are the real manifolds of a unique complex string that lives on the complex two sphere:

$$Y^{(2)}(t, s; k') = Y^{(1)}(it, is; k) \quad (59)$$

Putting everything together we finally obtain the string represented in Figure 5.

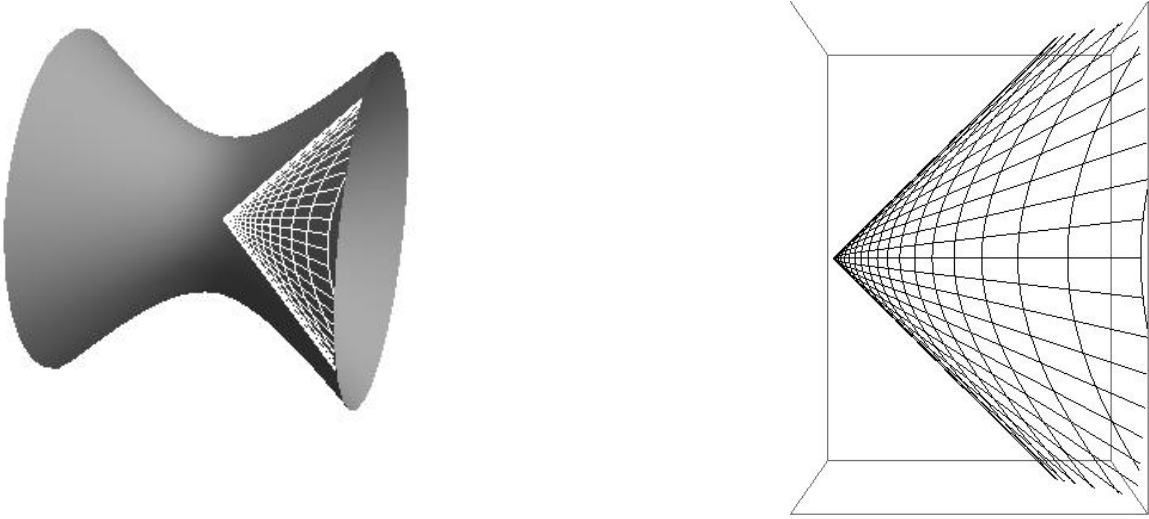


Figure 4: The anti de Sitterian semi-infinite string (55) extending towards the spacelike boundary.

7 Euclidean strings.

The above strings exhaust the possible open strings on the two dimensional anti de Sitter (resp. de Sitter) manifold. What about the two remaining possible quotients? They will not give rise to any new solution. Indeed let us single out the second coordinate $\xi_1(\hat{t}, \hat{s})$; we get

$$(t, s) \rightarrow \begin{cases} Y_0(t, s) &= \frac{\xi_2}{\xi_1} = k \operatorname{cd}(t, k) \operatorname{cd}(s, k) &= k \operatorname{sn}(t + K, k) \operatorname{sn}(s + K, k), \\ Y_1(t, s) &= \frac{\xi_3}{\xi_1} = k' \operatorname{nd}(t, k) \operatorname{nd}(s, k) &= \frac{1}{k'} \operatorname{dn}(t + K, k) \operatorname{dn}(s + K, k), \\ Y_2(t, s) &= \frac{\xi_0}{\xi_1} = k' k \operatorname{sd}(t, k) \operatorname{sd}(s, k) &= \frac{k}{k'} \operatorname{cn}(t + K, k) \operatorname{cn}(s + K, k), \end{cases} \quad (60)$$

i. e. this solution is obtained from (44) by a quarter period shift.

$$Y^{(3)}(t, s; k') = Y^{(1)}(t + K, s + K; k) \quad (61)$$

Similarly the solution obtained by singling out the first coordinate ξ^0 coincides with (55) by a quarter period shift:

$$Y^{(4)}(t, s; k') = Y^{(2)}(t + K, s + K; k). \quad (62)$$

As a final remark we notice that by replacing t with it in Eq. (60) we get that the embedding

$$(t, s) \rightarrow \begin{cases} X_0(t, s) &= \frac{1}{\sqrt{1-k^2}} \operatorname{dc}(t, k') \operatorname{dn}(s, k) = \frac{\vartheta_2}{\vartheta_4} \frac{\vartheta_3(\hat{t})}{\vartheta_2(\hat{t})} \frac{\vartheta_3(\hat{s})}{\vartheta_4(\hat{s})}, \\ X_1(t, s) &= \frac{k}{k'} \operatorname{nc}(t, k') \operatorname{cn}(s, k) = \frac{\vartheta_2}{\vartheta_4} \frac{\vartheta_4(\hat{t})}{\vartheta_2(\hat{t})} \frac{\vartheta_2(\hat{s})}{\vartheta_4(\hat{s})}, \\ X_2(t, s) &= k \operatorname{sc}(t, k') \operatorname{sn}(s, k) = \frac{\vartheta_2}{\vartheta_4} \frac{\vartheta_1(\hat{t})}{\vartheta_2(\hat{t})} \frac{\vartheta_1(\hat{s})}{\vartheta_4(\hat{s})}, \end{cases} \quad (63)$$

represents either a closed string expanding or an open string oscillating in a Lobachevsky space, identified here with the Euclidean AdS manifolds. Identical result is obtained by the replacement s with is . Thus our meethod provides also solution for the hyperbolic sigma-model that are real. On the other hand, as it is well known, this is not possible for the corresponding sigma-model on the sphere, subject to the constraints (35) [9].

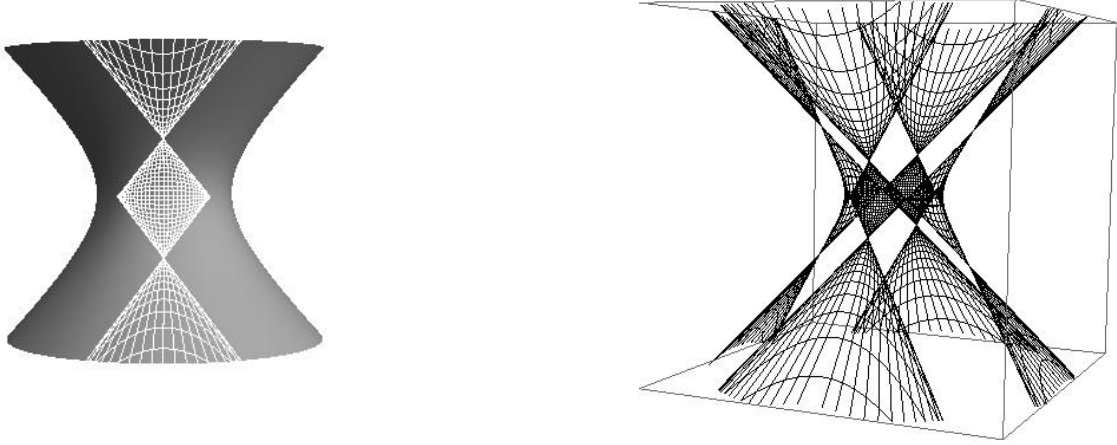


Figure 5: We represent here the complete solution in the de Sitter case. At the left a string contracting to a point, expanding and recontracting again, and then expanding forever. At the right the complete (real) solution. Every time that the string passes through a singular point the elliptic modulus jumps to the complementary modulus: the four patches contiguous to a given patch of elliptic modulus k all have modulus k' .

8 Open/closed self-dual strings

There is a second well-known relations between theta functions ([8], p 487) that may be mapped into solutions of the string equations on AdS_2 (or either dS_2):

$$\xi^2 = \vartheta_1(\hat{t})^2 \vartheta_3(\hat{s})^2 + \vartheta_2(\hat{t})^2 \vartheta_4(\hat{s})^2 - \vartheta_3(\hat{t})^2 \vartheta_1(\hat{s})^2 - \vartheta_4(\hat{t})^2 \vartheta_2(\hat{s})^2 = 0. \quad (64)$$

The associated two-surface embedded in the cone $C_{2,2}$ is now given by:

$$t, s \rightarrow \begin{cases} \xi_0 = \vartheta_1(\hat{t}) \vartheta_3(\hat{s}), & \xi_1 = \vartheta_2(\hat{t}) \vartheta_4(\hat{s}), \\ \xi_2 = \vartheta_3(\hat{t}) \vartheta_1(\hat{s}), & \xi_3 = \vartheta_4(\hat{t}) \vartheta_2(\hat{s}), \end{cases} \quad (65)$$

and the embedding

$$(t, s) \rightarrow \begin{cases} Y_0(t, s) &= \frac{\xi_0}{\xi_2} &= \text{sn}(t, k) \text{dc}(s, k), \\ Y_1(t, s) &= \frac{\xi_1}{\xi_2} &= \text{cn}(t, k) \text{nc}(s, k), \\ Y_2(t, s) &= \frac{\xi_3}{\xi_2} &= \text{dn}(t, k) \text{sc}(s, k) \end{cases} \quad (66)$$

represents a string. The anti-de Sitter interpretation is that of an infinite string vibrating periodically. No point moves at the speed of light and therefore the string extends to infinity. Exchanging the roles of t and s the above solution may also be seen as a closed string wrapping around a two-dimensional de Sitter spacetime and undergoing a (cosmological) process contraction followed by an expansion, while vibrating. The equal time section of the string are however not cosmological sections of the de Sitter manifold (see figure 6). The other possible ratios of the homogeneous coordinates do not produce any new solution as it may be understood by the absence of a boundary of the worldsheet in either the AdS or the dS case or circling around the manifold. For instance

$$(t, s) \rightarrow \begin{cases} Y_0(t, s) &= \frac{\xi_0}{\xi_2} &= \text{sd}(t, k) \text{ds}(s, k) = \text{cn}(t + K, k) \text{nc}(s + K, k), \\ Y_1(t, s) &= \frac{\xi_1}{\xi_2} &= \text{cd}(t, k) \text{ns}(s, k) = -\text{sn}(t + K, k) \text{dc}(s + K, k) \\ Y_2(t, s) &= \frac{\xi_3}{\xi_2} &= \text{nd}(t, k) \text{cs}(s, k) = -\text{dn}(t + K, k) \text{sc}(s + K, k) \end{cases} \quad (67)$$

and the solution is obtained from the previous simply by a quarter period shift in both the worldsheet coordinates.

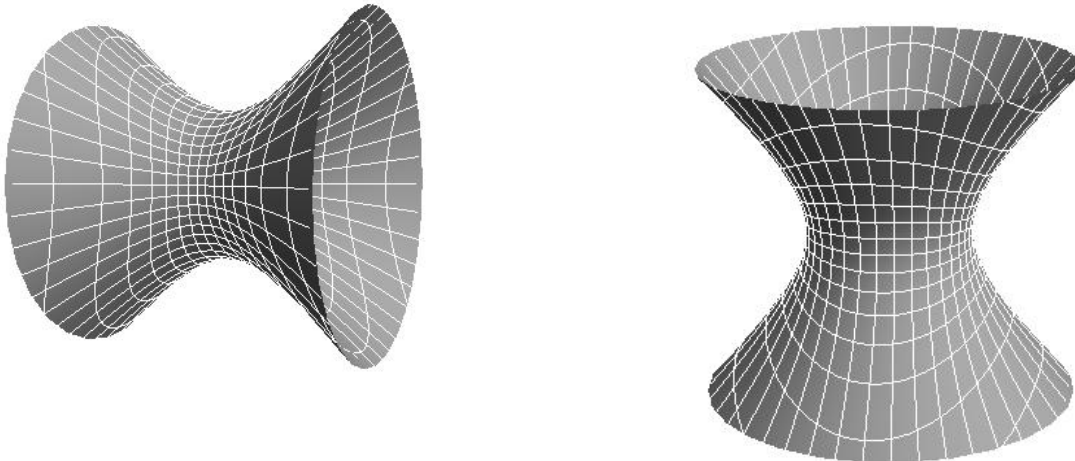


Figure 6: At the left: the infinite AdS solution. At the right: the solution wrapping around the de Sitter space. In both cases no point moves at the speed of light and the string extends all over the whole spacetime.

9 Conclusions and Outlook

We have presented a class of doubly elliptic solution of the string equations on the de Sitter and the Anti-de Sitter manifold. The main ingredients of our treatment are the separation of variables in the string equations and the relation of the model with the algebraic structure of the Jacobian elliptic theta functions. In the present context that relation is made clear through the embedding of the strings in the projective cone. The constant curvature manifold arise as quotient of that cone. In a forthcoming work [6] we will hopefully present a general treatment of the separation of variables not limited to the two-dimensional case as well as nontrivial solutions of the string equation.

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